How to get around local dependence

Norman Verhelst
National Institute for Educational Measurement (Cito)
Arnhem, The Netherlands

Reference

Verhelst, N.D. & Verstralen, H.H.F.M. (2008). Some Considerations on the Partial Credit Model. *Psicologica, 29,* 229-254.

Focus of the Research

- Mental processes
  - Theory of difficulty
  - Explaining interactions between items
- Measuring and comparing individuals or populations
  - Example: international surveys
  - Interactions are a nuisance

The Partial Credit Model (PCM)

(1) Andersen (1977): $P(X_i = j) \propto \exp(j\theta - \eta_{ij})$

(2) Masters (1982): $P(X_i = j) \propto \exp(j\theta - \sum_{g=0}^{j} \beta_{ig})$

(3) Reparameterization:

$$\eta_{ij} = \sum_{g=0}^{j} \beta_{ig} \text{ and } \beta_{ij} = \eta_{ij} - \eta_{i,j-1}, (j = 1, \ldots, m_i)$$

(4) $\eta_{i0} = \beta_{i0} = 0$

Some Facts and Fantasies

- The PCM is an exponential family
- The parameter space (per item) is $\mathbb{R}^m$
- The steps interpretation of the category parameters does not hold (Molenaar, 1983)
- The PCM is a polytomous generalization of the Rasch model for binary items, not the generalization
  - Graded response model of Samejima
  - Polytomous model of Rasch
  - Model of Tutz

The PCM and the Rasch model

- Assume the Rasch model is valid for k items.
- Consider a partition into T testlets
  - Number of items in testlet t is m_t (≥ 2)
- Testlet score = number of items correct for the testlet
- Suppose only testlet scores are observed

Distribution of testlet score S (1)

$$P(X_i = x_i \mid \theta) \propto \exp[x_i(\theta + \beta_i)], (i = 1, \ldots, m)$$

Reparameterization: $\varepsilon_i = \exp(\beta_i)$

$$P(S = s \mid \theta) \propto \exp(s\theta) \sum_{\sum z_i = s} \prod_{i=1}^{m} \varepsilon_i^{z_i}, (s = 0, \ldots, m; z_i \in \{0,1\})$$

Basic symmetric functions

$$\gamma_s \equiv \gamma_s(\varepsilon) = \sum_{\sum_i z_i = s} \prod_{i=1}^{m} \varepsilon_i^{z_i}, (s = 0, \ldots, m; z_i \in \{0,1\})$$

$$\gamma_0 = 1$$

$$\gamma_1 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$$

$$\gamma_2 = \varepsilon_1 \varepsilon_2 + \varepsilon_1 \varepsilon_3 + \varepsilon_2 \varepsilon_3$$

$$\gamma_3 = \varepsilon_1 \varepsilon_2 \varepsilon_3$$

Distribution of testlet score S (2)

$$P(S = s \mid \theta) \propto \exp(s\theta) \sum_{\sum z_i = s} \prod_{i=1}^{m} \varepsilon_i^{z_i} = \exp(s\theta) \times \gamma_s(\varepsilon)$$

$$\text{Define}: \eta_s = -\ln(\gamma_s)$$

$$P(S = s \mid \theta) \propto \exp(s\theta - \eta_s) \quad \Rightarrow \; \text{PCM}$$

Proposition

- If items follow the Rasch model, then testlet scores follow the PCM
- The relation between the parameters is given by $\eta_s = -\ln[\gamma_s(\varepsilon)] \quad (*)$
- But the converse is not generally true:

  If it were true, then for *m* arbitrary numbers $\eta_1,\ldots,\eta_m$ there would be *m* positive real numbers $\varepsilon_1,\ldots,\varepsilon_m$ such that (*) is fulfilled

Polynomials

$$\text{Polynomial in } x: P_m(x) = \prod_{i=1}^{m} (x + \varepsilon_i)$$

$$P_m(x) = \gamma_0 x^m + \gamma_1 x^{m-1} + \gamma_2 x^{m-2} + \cdots + \gamma_m x^0$$

Theorem 1 (Isaac Newton)

- If a polynomial $P_m(x)$ has real coefficients $\gamma_0, \dots, \gamma_m$, then, if there are m real roots, it holds that

$$\frac{(s+1)(m-s+1)}{s(m-s)}\gamma_{s-1}\gamma_{s+1} \le \gamma_s^2, \quad (s = 1, \dots, m-1)$$

Necessary, not sufficient

- Example: $m = 3$
- Coefficients: 1, 9, 25 and 17
- Fulfill Newton's conditions
- Roots are: -1, -4+i and -4-i

Results thus far

- Testlets formed from Rasch items cover partly but not completely the parameter space of the PCM.
- Rasch model assumes local independence
- Do we find full coverage if we allow for dependence (within testlets) ?

Sets of ordered *g*-tuples

$$I_1 = \{1, 2, \ldots, m\}$$

$$I_2 = \{(1, 2), (1, 3), \ldots (1, m), (2, 3), \ldots, (2, m), \ldots, (m-1, m)\}$$

$$I_3 = \{(1, 2, 3), (1, 2, 4), \ldots, (m-2, m-1, m)\}$$

$$\cdots$$

$$I_m = \{(1, 2, \ldots, m)\}$$

The dependence model

$$P(\underline{X} = \underline{x} \mid \theta) \propto \exp(s\theta) \times$$

$$\exp\left[\sum_{i \in I_1} x_i \beta_i + \sum_{(i,j) \in I_2} x_i x_j \beta_{ij} + \cdots + \sum_{I_m} x_1 x_2 \cdots x_m \beta_{ij\cdots m}\right]$$

The Rasch model is a special case

Submodels studied by Kelderman and Rijkes

Distribution of testlet score S (1)

$$P(S = s \mid \theta) \propto \exp(s\theta) \times$$

$$\sum_{\sum_i z_i = s} \exp\left[\sum_{i \in I_1} z_i \beta_i + \sum_{(i,j) \in I_2} z_i z_j \beta_{ij} + \cdots + \sum_{I_m} z_1 z_2 \cdots z_m \beta_{ij\cdots m} \right]$$

Reparameterization :

$$\varepsilon_i = \exp(\beta_i); \quad \varepsilon_{ij} = \exp(\beta_{ij}); \cdots ; \varepsilon_{ij\cdots m} = \exp(\beta_{ij\cdots m})$$

$$\underline{\varepsilon}^* = (\varepsilon_1, \ldots, \varepsilon_m, \varepsilon_{12}, \ldots, \varepsilon_{m-1,m}, \ldots, \varepsilon_{12\cdots m})$$

$$\dim(\varepsilon^*) = 2^m - 1$$

Distribution of testlet score S (2)

$$P(S=s\,|\,\theta) \propto \exp(s\theta) \times$$

$$\sum_{\sum_i z_i = s} \exp\left[\sum_{i\in I_1} z_i\beta_i + \sum_{(i,j)\in I_2} z_i z_j \beta_{ij} + \cdots + \sum_{I_m} z_1 z_2 \cdots z_m \beta_{ij\cdots m}\right] =$$

$$\exp(s\theta) \times \sum_{\sum_i z_i = s}\left[\prod_i \varepsilon_i^{z_i} \times \prod_{I_2} \varepsilon_{ij}^{z_i z_j} \times \cdots \times \prod_{I_m} \varepsilon_{12\cdots m}^{z_1 z_2 \cdots z_m}\right] =$$

$$\exp(s\theta) \times \Gamma_s(\underline{\varepsilon}^*)$$

Example for *m*=3

$$\Gamma_0(\underline{\varepsilon}^*) = 1$$

$$\Gamma_m(\underline{\varepsilon}^*) = \prod_{g=1}^{2^m - 1} \varepsilon_g^*$$

Score = 2				Score =1			
pattern			term	pattern			term
1	1	0	$\varepsilon_1\varepsilon_2\varepsilon_{12}$	1	0	0	ε_1
1	0	1	$\varepsilon_1\varepsilon_3\varepsilon_{13}$	0	1	0	ε_2
0	1	1	$\varepsilon_2\varepsilon_3\varepsilon_{23}$	0	0	1	ε_3

Distribution of testlet score S (3)

$$P(S = s \mid \theta) \propto \exp(s\theta) \times \Gamma_s(\underline{\varepsilon}^*)$$

$$\text{Define}: \eta_s = -\ln[\Gamma_s(\varepsilon^*)]$$

$$P(S = s \mid \theta) \propto \exp(s\theta - \eta_s) \quad \Rightarrow \quad \text{PCM}$$

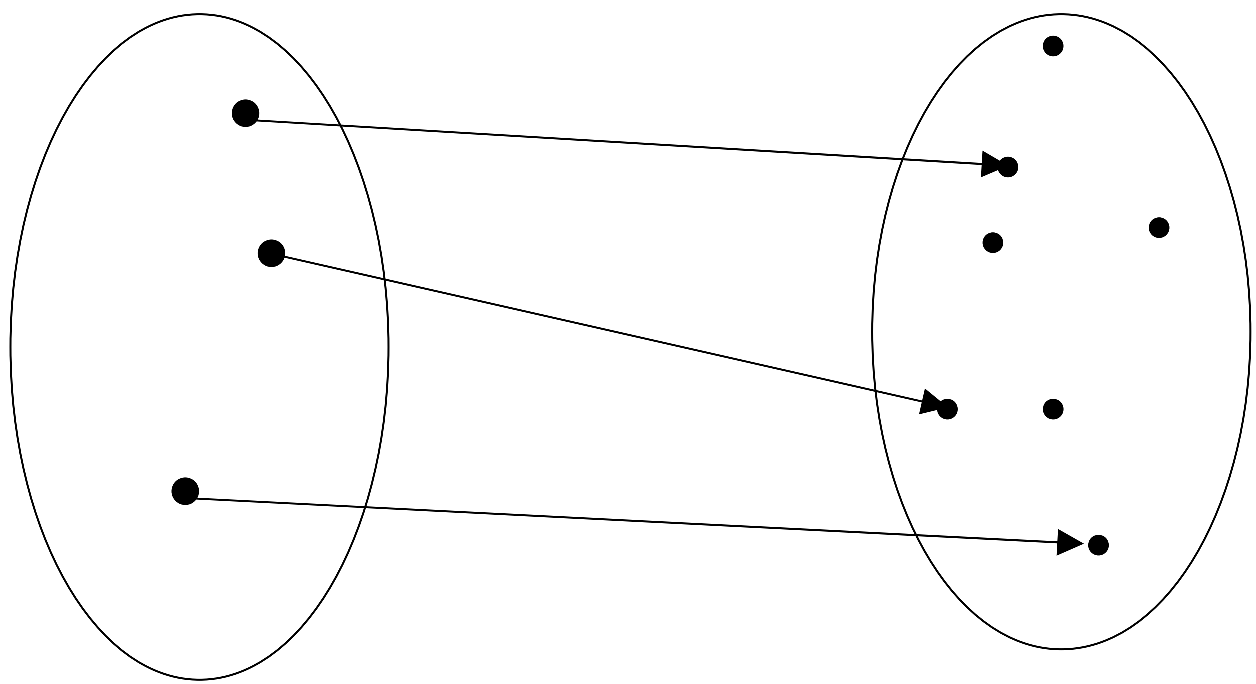
Theorem 2

Dependence model

PCM

dimension = 2^m-1

dimension = m



An *m*-dimensional subspace

$$\text{Consider the restricted model:}$$

$$\beta_h = \lambda_g \text{ for all } h \in I_g, (g = 1, \ldots, m)$$

$$\beta_1 = \beta_2 = \cdots = \beta_m = \lambda_1$$

$$\beta_{12} = \beta_{13} = \cdots = \beta_{m-1,m} = \lambda_2$$

$$\beta_{123} = \beta_{124} = \cdots = \beta_{m-2,m-1,m} = \lambda_3$$

$$\ldots$$

$$\beta_{12\cdots m} = \lambda_m$$

Distribution of S in the restricted model

$$P(S=s\,|\,\theta) \propto \exp(s\theta)\times$$

$$\sum_{\sum_i z_i = s} \exp\left[\sum_{i\in I_1} z_i\beta_i + \sum_{(i,j)\in I_2} z_i z_j \beta_{ij} + \cdots + \sum_{I_m} z_1 z_2 \cdots z_m \beta_{ij\cdots m}\right]$$

$$P_R(S=s\,|\,\theta) = \exp\left[s\theta + \sum_{g=1}^{s} \binom{m}{g} \lambda_g\right]$$

Bijection f: $\lambda \rightarrow \eta$

$$\eta_1 = -m\lambda_1$$

$$\eta_2 = -m\lambda_1 - \binom{m}{2}\lambda_2$$

$$\cdots$$

$$\eta_s = -\sum_{g=1}^{s}\binom{m}{g}\lambda_g$$

$$\lambda_1 = -\frac{\eta_1}{m}$$

$$\lambda_2 = \frac{-\eta_2 - m\lambda_1}{\binom{m}{2}}$$

$$\cdots$$

$$\lambda_s = \frac{-\eta_s - \sum_{g=1}^{s-1}\binom{m}{g}\lambda_g}{\binom{m}{s}}$$

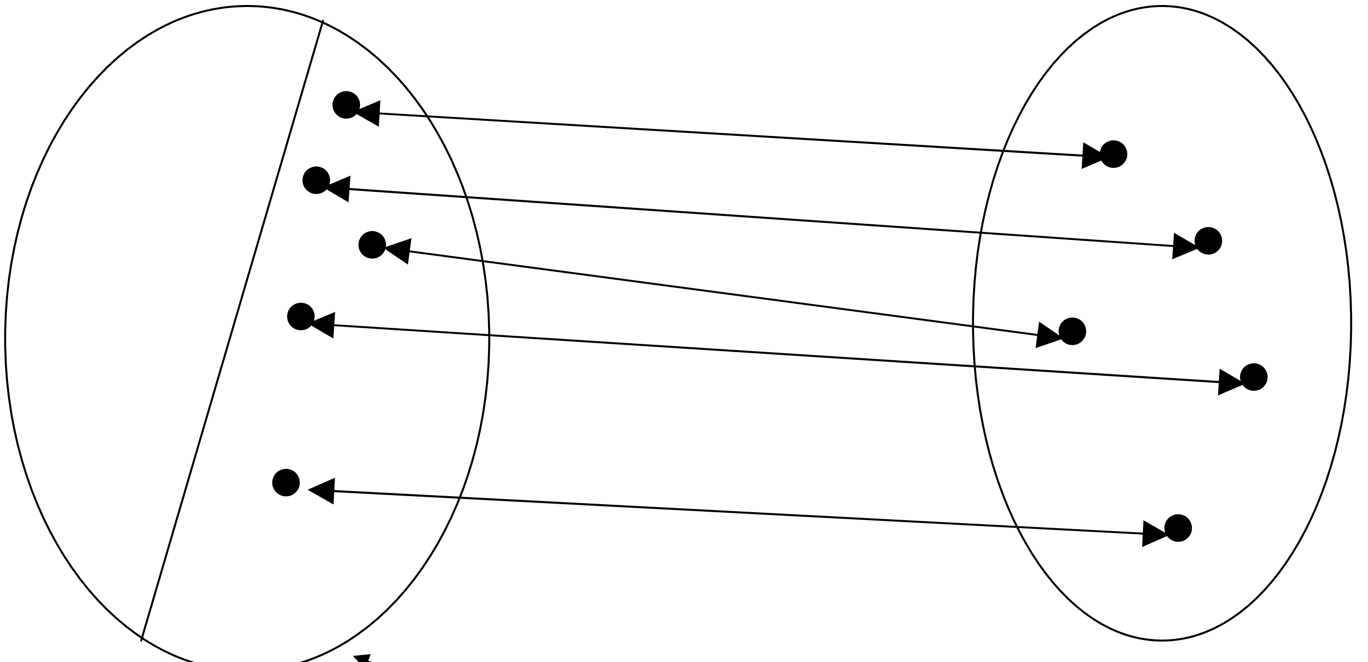
Theorem 3

Dependence model

dimension = 2^m-1

PCM

dimension = m



dimension = m

Example of a matching task with four items

Country		Capital
Belgium		Brussels
Estonia		Reykjavik
Iceland		Sofia
Albania		Tallinn
		Tirana

Practical consequences

- The restricted model induces a partition of the parameter space of the dependence model
  - From a PCM analysis of testlet scores, the dependence model cannot be identified.
  - The PCM can be interpreted as modelling the sum (within testlets) of the scores on binary items
- The PCM-parameters ‘absorbe’ all interaction effects
- No loss of information on the latent variable, since the score is sufficient for it

Some caveats

- The PCM-application assumes local independence between testlets
- The definition of testlets is vague: a partition of the original items
  - Items within the same task
  - Others; e.g., similarity between items
- What if the whole test is one testlet?

Thank you

Steps interpretation (1)

two-steps item: $\dfrac{1/2+0.25}{0.03}=?$

Version A: $\dfrac{1/2+0.25}{0.03}+1$

Version B: $\sqrt{\dfrac{1/2+0.25}{0.03}}$

Version C: $\dfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(x-\dfrac{1/2+0.25}{0.03})\exp(-x^2/2)dx$

Steps interpretation (2)

Response probabilities at $\theta = \theta_0$

score	0	1	2	3
Version B	0.1	0.45	0.15	0.3
Version C	0.1	0.45	0.44999	0.00001

$$\text{B: } P(X = 2 \mid X = 1 \text{ or } 2,\ \theta_0) = \frac{0.15}{0.15 + 0.45} = 0.25$$

$$\text{C: } P(X = 2 \mid X = 1 \text{ or } 2,\ \theta_0) \approx \frac{0.45}{0.45 + 0.45} = 0.5$$

Steps interpretation (3)

B: $\theta_0 - \beta_2 = \text{logit}(0.25) = -\ln 3 \Rightarrow \beta_2 = \theta_0 + \ln 3$

C: $\theta_0 - \beta_2 = \text{logit}(0.5) = 0 \Rightarrow \beta_2 = \theta_0$