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DYNAMIC PARAMETER ESTIMATION IN STUDENT MONITORING SYSTEMS

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Abstract

The Elo rating system has it origins in chess ability estimation. The system is proposed as a means of monitoring ability in progress testing. Data from progress tests often consists of varying numbers of measurements at varying time points. Applying the Elo rating system to such data has several advantages. There is no need for a growth model: both person ability and item difficulty are estimated simultaneous, an individual rating is updated immediately after a response, and the algorithm requires no computationally intensive calculations. Several theoretical properties of the Elo rating system are evaluated, such as stationarity of the mean, the variance and the distribution. The evaluation of these properties leads to several variants of the system. A logistic, normal ogive, and mixture variant are discussed. Variants using adaptive opponent matching are evaluated as well. The research concludes with an adaptive mixture variant of the Elo rating system in order to obtain a stationary distribution.

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1. Introduction

The field of educational measurement is increasingly being asked to provide the means to follow the progress of students over time. Such systems are known as *student monitoring systems*. In the Netherlands, examples include progress testing at the University of Maastricht's medical faculty, and in the psychology program at Erasmus University Rotterdam. International examples include the University of Missouri Medical School in Kansas City, and McMaster University in Hamilton, Canada. Student progress in a number of skills is also being monitored in primary and special education by testing systems developed by Cito, the Dutch national institute for test development. Interest in this topic stems from the possibilities to track individual growth and to compare individuals with their year group. Consequently, remedial teaching or accelerated programs can be offered to particular students.

In progress testing, both the frequency of test administration and the accuracy at each administration are important. Frequency of administration is important because it allows for quick intervention when atypical growth patterns are observed. Accuracy is usually obtained by administering longer tests and is especially important in high stakes testing. A concern in progress testing is to balance these two interests, the frequency at which tests are administered versus the test length with the response burden as the main constraint.

In monitoring abilities, two main traditions exist. One is to model change using growth models, the second approach is to track change. This report focuses on tracking changes, since specific growth patterns are not always plausible nor available. A popular tracking system is the *Elo rating system*. It is applied extensively in rating chess performances, where it was originally designed for by Arpad Elo in 1960. Other large scale applications are found in sports, computer games, and recently in the field of educational measurement. In sports and games often two players, or teams, compete, resulting in a win, loss or draw. Data from such matches are known as *paired comparisons*, for which the Elo rating system was designed. In the field of educational measurement, one can regard a player as a respondent and its opponent as an item. This approach allows for high frequency testing, by updating estimates after each item administered, instead of after a more lengthy test. It is noted that such frequent updates are not feasible using maximum likelihood estimation.

Since the Elo rating system has already been functioning for a long time in several large-scale implementations, it seems to work quite well in practice. However, it is not clear whether its basic properties allow for applications in educational measurement. The goal of this research is to investigate whether adopting Elo's algorithm in educational measurement provides us with correct parameter estimates. Several basic properties of the rating sytem were evaluated for this purpose.

2. Methods

2.1. **Elo rating system.** In 1960, Arpad Elo introduced a rating system designed to evaluate performances of chess players for the United States Chess Federation. Elo described his own rating system as follows:

> Simply stated, the Elo Rating System is a numerical system in which differences in rating may be converted into scoring or winning probabilities . . . It is a scientific approach to the evaluation of chess performances. (Elo, 1978, p. 16).

The rating system consists of several different forms with corresponding formulas, with different update formulas for the different types of matches. For example for rating a Round Robin, for periodic measurement and formulas with linear approximations for ease of calculations. In this paper we focus only on the Current Rating Formula for Continuous Measurement, which is cited below (Elo, 1978, p. 25). Further references to Elo's rating formula concern this specific instance.

$$R_n = R_o + K(W - W_e) \tag{1}$$

> R_n is the new rating after the event.
> R_o is the pre-event rating.
> K is the rating point value of a single game score.
> W is the actual game score, each win counting 1, each draw $1/2$.
> W_e is the expected game score based on R_o.

The player's rating is updated after every match, thus allowing for continuous measurement. The logic of the formula consists of a player gaining points if performing above his expectancy, and losing points if performing below his expectancy. Consequently, a player competing a much higher rated opponent risks losing few points when losing the game, with the possibility of gaining many points when winning. The opposite holds for the high rated player against a lower rated opponent.

The *Elo Rating System* (ERS) is a formula for rating pairwise comparisons. At the root of the ERS lies the *Bradley Terry Luce* (BTL) model, providing the expected outcome W_e in Formula (1). The BTL model is closely related to the *Rasch Model* (RM) from the field of educational measurement, where in the BTL a person opposes another person instead of an item (e.g., Rasch, 1960; Bradley & Terry, 1952). The probability that person j wins is given by

$$P(Y_{jk} = 1) = \frac{\exp(\theta_j - \theta_k)}{1 + \exp(\theta_j - \theta_k)} \quad (2)$$

where $Y_{jk} = 1$ represents a win for person j as the outcome of a match between person j and k, with abilities θ_j and θ_k respectively. When both persons are equally able, it follows that the probability of person j or k winning is $1/2$. If the ability of person j is larger than that of person k, the probability of person j winning is larger than $1/2$. It is noted that in the current research, possible match outcomes are a win or a loss; the possibility of a draw in the context of testing is left open for further research.

The ERS has several large-scale implementations, of which the most well known are the ratings of the *World Chess Federation* (FIDE) and several national chess federations. A short simulation is illustrated in Figure (1) to show how the rating updates of the system work. In this simulation,

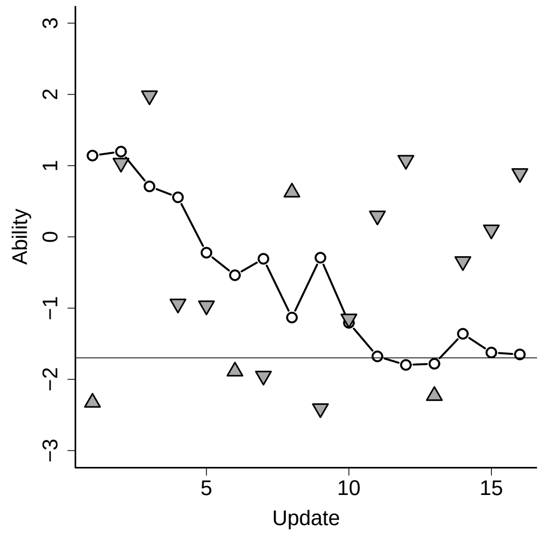

FIGURE 1. Detailed simulation of ability estimation

true person ability and true item difficulties are sampled from a normal distribution. The figure presents the ability estimate of a single person whose true ability is indicated by a horizontal line. The BTL model is used for calculating winning probabilities. The respondent's ability estimate is followed over time, while he is assigned to a random selection of a number of items. The vertical position of the triangles in the figure indicate the difficulty of the items, where the direction of the triangle indicates winning (pointing up) or losing (pointing down). It is noted that after winning from an easy item, the respondent's ability estimate improves little, while after winning from a difficult item the ability improves considerably. The opposite holds for losing. Another observation is that the estimated ability is updated with a certain step size, and that the algorithm therefore maintains an *intrinsic noise*, e.g., the estimate hops up and down around its true value. Figure (2) illustrates these properties in a longer run, where a somewhat longer simulation illustrates how the estimate is in time corrected in the direction of its true value, even if the starting value is quite off.

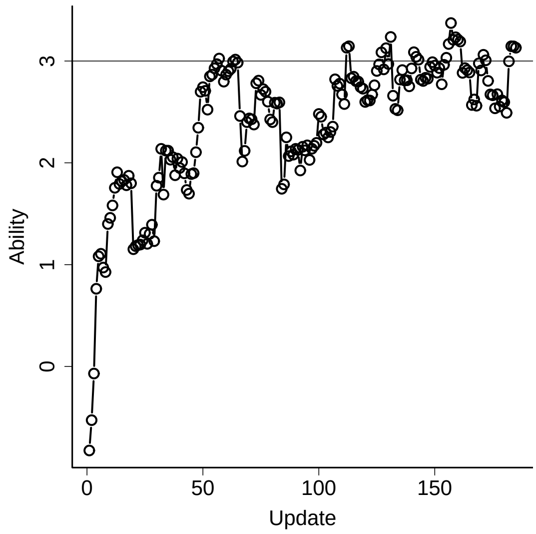

FIGURE 2. Long simulation of ability estimation

Many extensions of the ERS are possible. For example, different update rules can be evaluated, the intrinsic noise contained in the estimate can be dealt with by using smoothing methods, or standard errors can be estimated along. In the ERS as applied in chess, also many ad-hoc methods are present to control or enhance its characteristics, like dealing with rating inflation, isolated competitions, and so on. While such extensions are useful, for applications in educational measurement a further inspection of the basic properties of the algorithm is necessary. Comparable extensions are for that reason not considered. For further reading concerning such extensions, one is referred to the rating system developed by Mark Glickman (e.g., Glickman, 1999, 2001).

In this paper, the basic properties of several variants of the ERS are discussed sequentially. First a *non-adaptive logistic ERS* is evaluated. It is non-adaptive, since there is the possibility of choosing an opponent freely and it is logistic since a logistic response function is used. Second, a *non-adaptive* *normal ogive model is evaluated*, where the logistic response function is replaced with a normal ogive response function. Next, adaptive versions of these two variants are reviewed, where an item or opponent selection step makes the system adaptive. Finally, properties of several solutions are combined, leading to a *mixture model*.

2.2. **Non-adaptive logistic Elo rating system.** A simple situation is considered in which two competitors, or a single respondent and a single question, only compete against each other with static abilities (e.g., no learning is assumed). It is assumed in this simulation that the Bernoulli distributed responses Y at occasion i follow a logistic response function. Concerning the rating updates, a simplified notation of the ERS is used. Instead of estimating ability for both players, the ability *difference* between the two players is estimated, presented in the formula below.

$$P(Y_i = 1|\theta) = \frac{\exp(\theta)}{1+\exp(\theta)} = \mu(\theta)$$

where $\theta = \theta_i - \theta_j$ represents the true ability difference between the two players, being constant in the current simulation.

The ERS updates its ability difference estimate after every trial i, where X_i indicates the estimated ability difference between the players. The dichotomous variable Y_i indicates a win or a loss, and the factor K is a multiplicand of the update step size. The ERS is then represented by

$$X_i = X_{i-1} + K\left[Y_i - \mu\left(X_{i-1}\right)\right].$$

where the expected winning probability μ is modeled with the BTL model

$$\mu(x_{i-1}) = \frac{\exp(x_{i-1})}{1+\exp(x_{i-1})}.$$

One should note how each update only requires the previous ability estimate X_{i-1}, the scaling factor K and the outcome of the match Y_i. The history of performances is only represented by the previous rating. The rating scheme satisfies the Markov property, but maintains intrinsic noise. The estimate itself will not converge to the true value but its mean should. The first property of this Markov chain that is inspected, is whether the mean converges to a constant over time, thus whether stationarity of the mean holds.

2.2.1. *Stationarity of the mean.* An approach in which two steps of the algorithm are written out to illustrate stationarity of the mean can be found in Appendix A. In general, for the mean to be stationary the expected value of mean of the next time point should be equal to that of the previous. In the following equation, this stationarity property will be rewritten to determine a solution.

$$\begin{aligned}
\mathcal{E}(X_i) &= \mathcal{E}(X_{i-1}) \\
&\Updownarrow \text{ where the ERS can be filled in for } X_i \\
\mathcal{E}(X_{i-1}) &= \mathcal{E}\left\{X_{i-1} + K\left[Y_i - \mu(X_{i-1})\right]\right\} \\
&\Updownarrow \text{ rewriting the stationarity condition} \\
\mathcal{E}\left\{X_{i-1} + K\left[Y_i - \mu(X_{i-1})\right]\right\} &= \mathcal{E}(X_{i-1}) \\
&\Updownarrow \text{ which can be separated into parts} \\
K\left\{\mathcal{E}(Y_i|\theta) - \mathcal{E}\left[\mu(X_{i-1})\right]\right\} &= 0
\end{aligned}$$

Stationarity about the mean is assured if this last condition holds, which is true for $K = 0$ or if the following is true

$$\begin{aligned}
\mathcal{E}(Y_i|\theta) &= \mathcal{E}[\mu(X_{i-1})] \\
&\Updownarrow \\
\frac{\exp(\theta)}{1+\exp(\theta)} &= \mathcal{E}\left[\frac{\exp(X_{i-1})}{1+\exp(X_{i-1})}\right]
\end{aligned} \tag{3}$$

Equation (3) is written as an equality, but is problematic to satisfy. Figure (3) illustrates the discrepancy between these functions. The dashed lines

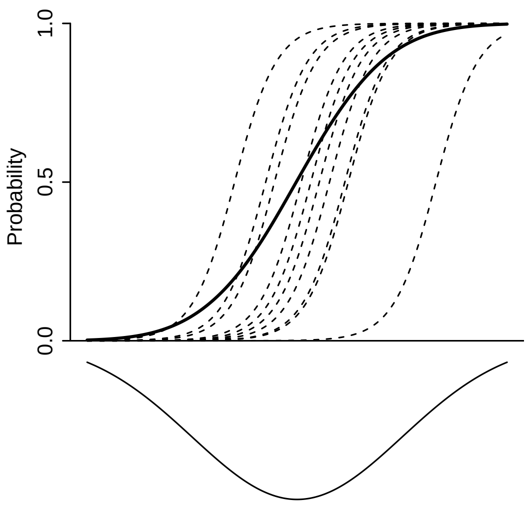

FIGURE 3. Cumulative distribution

represent the different X_{i-1} response functions, which are sampled from the distribution at the bottom of the figure. The thicker line is their expected value. One can observe how the variance in X_{i-1} causes the slope of the average response function to decrease, creating the difference with the expit of the true ability θ. A slope correction is therefore needed, which is introduced later in equation (4). First, another illustration is provided in Figure

(4), with an exaggerated scaling factor $K = 10$ to demonstrate the bias in the estimates due to the problematic equality presented earlier. The bold lines represent the upper and lower bounds of the intrinsic noise, with the thin line as average. Since the bias of this basic ERS can only be solved with a

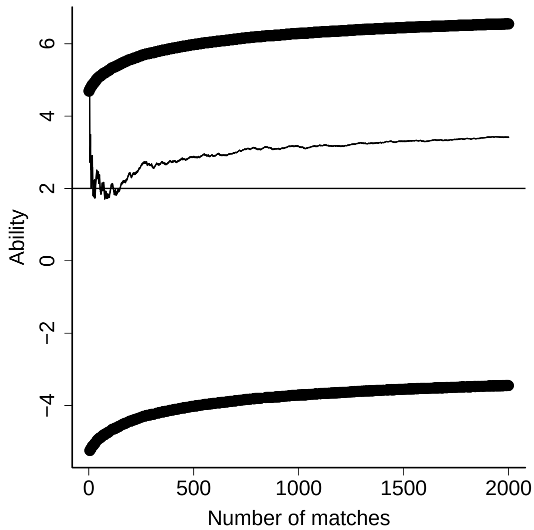

FIGURE 4. Simulation of biased ability estimation

step size of $K = 0$ or an implausible distribution of θ, alternative approaches are needed. Next, a normal ogive non-adaptive ERS is evaluated.

2.3. **Non-adaptive normal ogive Elo rating system.** An alternative to the logistic ERS is the normal ogive ERS, where the logistic *cumulative distribution function* (CDF) is replaced by the normal ogive CDF. Again, it was decided to model the difference between ability of two players playing each other without learning, by θ. The probability of winning is then

$$P(Y_i = 1|\theta) = \phi(\theta).$$

Following, the normal ogive ERS then takes on the following shape

$$X_i = X_{i-1} + K\left[Y_i - \phi\left(aX_{i-1}\right)\right] \tag{4}$$

where K is the regular scaling factor as described earlier, and a an added constant scaling factor to correct for the flattened slope.

2.3.1. *Stationarity of the mean.* As was done earlier for the logistic non-adaptive ERS, stationarity of the mean was investigated for the normal ogive ERS. Since the calculations are similar to those obtained for the logistic variant, the arguments below are condensed

$$\begin{aligned} \mathcal{E}(X_i) &= \mathcal{E}(X_{i-1}) \\ &\Updownarrow \\ \mathcal{E}(Y_i) &= \mathcal{E}\left[\phi(aX_{i-1})\right] \\ &\Updownarrow \\ \phi(\theta) &= \mathcal{E}\left[\phi(aX_{i-1})\right]. \end{aligned} \tag{5}$$

The last condition must hold to assure stationarity of the mean. In the logistic ERS, this last equation was hard to satisfy as can be seen in equation (3). If one assumes a normal distributed X_{i-1}, a proof is provided by Lord and Novick (1968, p.377) in equation (6). While this assumption might not be very plausible to hold in practice, it is chosen to facilitate further calculations.

$$\phi\left(\frac{\theta}{\sqrt{\sigma^2 + \frac{1}{a^2}}}\right) = \mathcal{E}\left[\phi(aX_{i-1})\right] \tag{6}$$

To equate the proof in equation (6) with the condition for stationarity of the mean in equation (5), the following must hold

$$\begin{aligned} \phi(\theta) &= \phi\left(\frac{\theta}{\sqrt{\sigma^2 + \frac{1}{a^2}}}\right) \\ &\Updownarrow \quad \text{which holds if} \\ \sqrt{\sigma^2 + \frac{1}{a^2}} &= 1 \\ &\Updownarrow \quad \text{rewrite as depending on } a \\ a &= \frac{1}{\sqrt{1-\sigma^2}}, \quad \sigma^2 < 1 \end{aligned}$$

Thus a solution is obtained in which the scaling factor a is adjusted to solve the equation above. With this condition satisfied, stationarity of the mean is obtained with a rather strong assumption. Since not only the mean but also the variance of the distribution should be stationary, this property is investigated next.

2.3.2. *Stationarity of the variance.* The variance of the occasions can be derived in the same way as the stationarity of the mean. The variance at time point $i-1$ was equated with the variance at time point i

$$var(X_i) = var(X_{i-1})$$

which can be rewritten using theorem (1) as a sum of variances and a covariance.

Theorem 1 (Variance of a sum of variables)**.** *The variance of a sum of variables can be decomposed into a sum of variances and a covariance*

$$var(aX + bY) = a^2 var(X) + b^2 var(Y) + 2ab \cdot cov(X, Y).$$

Proof.

$$\begin{aligned} var(aX + bY) &= cov(aX + bY, aX + bY) \\ &= a^2 var(X) + b^2 var(Y) + 2ab \cdot cov(X, Y) \end{aligned}$$

□

The following equation is obtained by applying this theorem and filling in the normal ogive ERS for X_i. Note that since this algorithm is not adaptive, the (un)correct answer to an item does not depend on the estimated ability, but on true ability instead: $cov(Y_i, X_{i-1}) = 0$ and $cov\left[Y_i, \phi(aX_{i-1})\right] = 0$.

$$\begin{aligned} var(X_i) &= var(X_{i-1}) + K^2 var(Y_i) + K^2 var[\phi(aX_{i-1})] \\ &\quad -2K cov\left[X_{i-1}, \phi(aX_{i-1})\right] \\ &\Updownarrow \text{ which leads to the following condition} \\ 0 &= K^2 var(Y_i) + K^2 var[\phi(aX_{i-1})] \\ &\quad -2K cov\left[X_{i-1}, \phi(aX_{i-1})\right] \\ &\Updownarrow \text{ separate } K \\ 2cov\left[X_{i-1}, \phi(aX_{i-1})\right] &= K\left\{var(Y_i) + var[\phi(aX_{i-1})]\right\} \\ &\Updownarrow \text{ obtain a solution with respect to } K \\ K &= 2\frac{cov\left[X_{i-1}, \phi(aX_{i-1})\right]}{var(Y_i) + var[\phi(aX_{i-1})]} \end{aligned}$$

A solution is obtained where the variance is stationary with K depending on two variances and a covariance. First, the covariance will be dealt with. The covariance between X_{i-1} and $\phi(aX_{i-1})$ can be obtained by differentiating

$\mathcal{E}\left[\phi(aX_{i-1})\right]$ with respect to θ.

$$\begin{aligned}\mathcal{E}\left[\phi(aX_{i-1})\right] &= \int_{-\infty}^{\infty}\phi(ax)\frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{(x-\theta)^2}{2\sigma^2}\right]dx\\ &\Updownarrow\\ \frac{\partial}{\partial\theta}\mathcal{E}\left[\phi(aX_{i-1})\right] &= \int_{-\infty}^{\infty}\phi(ax)\frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{(x-\theta)^2}{2\sigma^2}\right]\frac{2(x-\theta)}{2\sigma^2}dx\\ &= \frac{1}{\sigma^2}\left\{\int_{-\infty}^{\infty}\phi(ax)\frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{(x-\theta)^2}{2\sigma^2}\right]xdx-\theta\mathcal{E}\left[\phi(aX_{i-1})\right]\right\}\\ &= \frac{1}{\sigma^2}\left\{\mathcal{E}\left[X_{i-1}\phi(aX_{i-1})\right]-\theta\mathcal{E}\left[\phi(aX_{i-1})\right]\right\}\end{aligned}$$

and since

$$cov\left[X_{i-1},\phi(aX_{i-1})\right]=\mathcal{E}\left[X_{i-1}\phi(aX_{i-1})\right]-\mathcal{E}\left(X_{i-1}\right)\mathcal{E}\left[\phi(aX_{i-1})\right]$$

and

$$\mathcal{E}(X_{i-1})=\theta$$

resulting in

$$\begin{aligned}\frac{\partial}{\partial\theta}\mathcal{E}\left[\phi(aX_{i-1})\right] &= \frac{cov\left[X_{i-1},\phi(aX_{i-1})\right]}{\sigma^2}\\ &\Updownarrow\\ cov\left[X_{i-1},\phi(aX_{i-1})\right] &= \sigma^2\frac{\partial}{\partial\theta}\mathcal{E}\left[\phi(aX_{i-1})\right]\\ &= \sigma^2\frac{\partial}{\partial\theta}\mathcal{E}\left[\phi(\theta)\right]\\ &= \sigma^2\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{\theta^2}{\sigma^2}\right).\end{aligned}$$

Concerning the two variances, the variance of Y_i is obtained by rewriting the Bernoulli distribution as follows

$$\begin{aligned}var(Y_i) &= P(Y_i=1|\theta)\left[1-P(Y_i=1|\theta)\right]\\ &= \phi(\theta)\left[1-\phi(\theta)\right].\end{aligned}$$

The variance of $\phi(aX_{i-1})$ can be obtained if a is chosen such that $\phi(aX_{i-1})=\phi(\theta)$

$$\begin{aligned}var\left[\phi(aX_{i-1})\right] &= \mathcal{E}\left[\phi(aX_{i-1})^2\right]-\mathcal{E}\left[\phi(aX_{i-1})\right]^2\\ &= \mathcal{E}\left[\phi(aX_{i-1})^2\right]-\phi(\theta)^2\end{aligned}$$

where

$$\mathcal{E}\left[\phi(aX_{i-1})^2\right]=\int\phi(ax)^2\frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{(x-\theta)^2}{2\sigma^2}\right]dx$$

This formula however proves difficult to simplify further. In the following section we will therefore introduce a more tractable variant, where the *expected* probability of winning in the ERS update is held constant. In order to keep this probability constant, a different item needs to be selected depending on the current estimate of the ability. This approach leads to more tractable and manageable expressions.

2.4. **Adaptive Elo rating system.** The logistic and normal ogive response functions can be replaced by a more general response function F, as suggested by Batchelder, Bershad, and Simpson (1992). The response function is adaptive, since for every match played, an equally capable opponent or equally difficult item is selected, resulting in $\mathcal{E}(Y_i) = 1/2$. The assumption of matching equally able persons is discussed later, together with the assumption that the estimated difficulty of the competing items are known. The probability of winning using this response function F is noted below

$$P(Y_i = 1|\theta, X_{i-1} = x_{i-1}) = F(\theta - x_{i-1}).$$

Also for this variant, stationarity of the mean will be discussed next.

2.4.1. *Stationarity of the mean.* Stationarity can now be shown quite easily, condensing the intermediate steps presented earlier.

$$\begin{aligned}
\mathcal{E}(X_i) &= \mathcal{E}(X_{i-1}) \\
&\Updownarrow \text{ fill in the ERS for } X_i \\
\mathcal{E}(Y_i) &= 1/2 \\
&\Updownarrow \text{ fill in the response function } F \text{ for } Y_i \\
\mathcal{E}\left[F(\theta - X_{i-1})\right] &= 1/2, \qquad (7)
\end{aligned}$$

In theorem (2), it is shown that if X_{i-1} has a symmetric distribution around θ, stationarity about the mean is assured.

Theorem 2. *If distributions $F(x)$ and $G(x)$ conform to $F(x) = 1 - F(-x)$ and $G(x) = 1 - G(-x)$, then their densities are symmetric and $\int_{-\infty}^{\infty} F(x)g(x)dx = 1/2$.*

Proof. The expression can be simplified using the inverse CDF transformation, which can be found in Appendix D. □

Assuming X_i to be symmetrical around θ, stationarity of the mean is shown. X_i can be shown to be symmetrical since each answer pattern has an equally likely complement, leading to a value of X_i equally distant from θ. The value of X_i only depends on K, the starting value θ, and the number of correct and incorrect answers and can be denoted as follows

(8) $X_i = \theta + {}^K\!/_2\,(n_{\text{correct}} - n_{\text{incorrect}})$ where $i = n_{\text{correct}} + n_{\text{incorrect}}$

which shows symmetry around θ. The symmetry of X_i is illustrated in Figure (5), where one can see that for every path there exists an equally likely counterpart. Since $P(y_i = 1) = 1 - P(y_i = 0)$, observing, for example, response pattern $Y = (001)$ is as likely as observing $Y = (110)$. Since symmetry of X_i

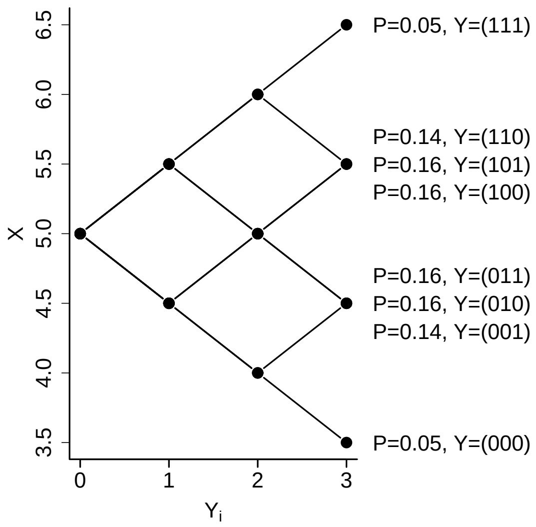

FIGURE 5. Symmetric probability tree

is shown, following from theorem (2), stationarity of the mean holds when starting at θ.

Since θ is unknown, a solution is needed to allow the algorithm to start at θ and hence to allow the mean to obtain stationarity. We can assume that when starting at an arbitrary value, the estimate will at some time pass the true value θ. We can consider all possible sample paths that reach θ at the same time point. For these paths, the argument presented above applies. A group of possible sample paths that reach θ at a specific time point are illustrated in Figure (6). All paths are shown that pass θ at $t = 5$ for the first time, after which stationarity of the mean is guaranteed. For every t, such a figure exists. Depending on the size of multiplicand K, the estimate X_i reaches θ with the passing of time t

$$f(x_{i=\infty}) = \int f(x_{i=\infty}|t) f(t)\,dt,$$

where $f(t)$ is the first passage time distribution. Judging from Figure (6), one can assume that this works. However, the exact properties of this first passage time distribution are left open as a suggestion for further research.

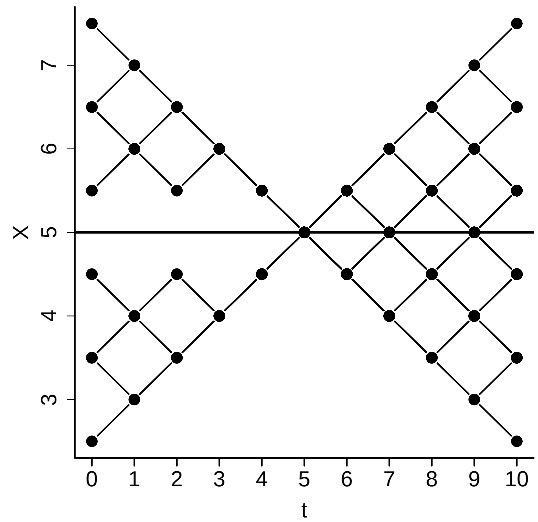

FIGURE 6. Passing time before stationarity of the mean

2.5. **Adaptive logistic Elo rating system.** In this section, for the adaptive ERS, a logistic response function is chosen as its symmetrical CDF, allowing the ERS to be written as follows

$$
\begin{aligned}
X_i &= X_{i-1} + K\,(Y_i - 1/2) \\
&\Updownarrow \\
X_i &= X_{i-1} + K\left[\frac{\exp(\theta - x_{i-1})}{1 + \exp(\theta - x_{i-1})} - \frac{1}{2}\right].
\end{aligned}
$$

It is noted that the update corrects itself in the direction of the true value, for example, when $X_{i-1} > \theta$

$$
\begin{aligned}
\mathcal{E}(X_i | X_{i-1} = x_{i-1}) &= X_{i-1} + K\left[\frac{\exp(\theta - X_{i-1})}{1 + \exp(\theta - X_{i-1})} - \frac{1}{2}\right] \\
&< X_{i-1}.
\end{aligned}
\tag{9}
$$

Owing to the fixed $\mu = 1/2$, the update step is $K/2$ in either direction. This leads to a decision tree-like structure, which is symmetric due to equal steps in either direction and has solutions laying on a grid due to $K/2$ size update steps. In section 2.4.1, stationarity of the mean is already shown for adaptive symmetric ERS in general, but is also written out for two update steps of the logistic variant in Appendix B.

2.5.1. *Stationarity of the variance.* While stationarity of the mean has already been shown for the general adaptive ERS, the stationarity of the variance should be considered specifically for the logistic variant. The variance of X_i can be written out as a sum of variances and a covariance, as stated in theorem (1).

$$\begin{aligned} var(X_i) &= var(X_{i-1} + KY_i) \\ &= var(X_{i-1}) + K^2 var(Y_i) + 2K cov(X_{i-1}, Y_i). \end{aligned} \tag{10}$$

One can recognize from equation (10) that stability of variance is achieved if

$$\begin{aligned} cov(X_{i-1}, Y_i) &= -K/2 \, var(Y_i) \\ &\Updownarrow \text{ insert in equation (10)} \\ var(X_i) &= var(X_{i-1}) + K^2 var(Y_i) + 2K \left[-K/2 \, var(Y_i)\right] \\ &\Updownarrow \; K \text{ drops out} \\ var(X_i) &= var(X_{i-1}). \end{aligned}$$

Since K drops out, the stationarity of the variance is independent of this scaling factor as long as $cov(X_{i-1}, Y_i) = -K/2 \, var(Y_i)$ holds. The next step consists of decomposing the last result using theorem (3) to replace Y_i with formula F.

$$\begin{aligned} cov(X_{i-1}, Y_i) &= \mathcal{E}\left[cov(X_{i-1}, Y_i | X_{i-1})\right] \\ &+ cov\left[\mathcal{E}(X_{i-1}|X_{i-1}), \mathcal{E}(Y_i|X_{i-1})\right] \\ &= cov\left[X_{i-1}, F(\theta - X_{i-1})\right]. \end{aligned} \tag{11}$$

Theorem 3 (Variance and covariance decomposition)**.** *If X and Y are random variables*

$$var(X) = \mathcal{E}\left[var(X|Y)\right] + var\left[\mathcal{E}(X|Y)\right]$$

and

$$cov(X, Y) = \mathcal{E}\left[cov(X, Y|X)\right] + cov\left[\mathcal{E}(X|X), \mathcal{E}(Y|X)\right]$$

Proof.

$$\begin{aligned} var(X) &= \mathcal{E}\left(X^2\right) - \mathcal{E}(X)^2 \\ &= \mathcal{E}\left[\mathcal{E}(X^2|Y)\right] - \mathcal{E}\left[\mathcal{E}(X|Y)^2\right] \\ &= \mathcal{E}[var(X|Y)] - var[\mathcal{E}(X|Y)] \end{aligned}$$

□

One can observe that the covariance $cov[X_{i-1}, F(\theta - X_{i-1})]$ includes X_{i-1} and a decreasing function of X_{i-1}. The covariance will therefore be negative. There might be a resulting positive value for K that will guarantee stationarity of the variance. Unfortunately, the results obtained are of the logistic normal form and are therefore quite difficult to solve. We will therefore further inspect the properties of the adaptive normal ogive ERS, which is more tractable.

2.6. **Adaptive normal ogive Elo rating system.** To find another approach to a solution for K, a situation can be considered where for the symmetric response function $F(\theta - X_{i-1})$ a normal CDF is chosen. It is assumed that $X_{i-1} \sim \mathcal{N}\left(\theta, \sigma^2\right)$, which is discussed in section 2.6.1. Equation (11) obtained in the previous section is rewritten below for the normal ogive variant.

$$\begin{aligned} cov\left[X_{i-1}, F(\theta - X_{i-1})\right] &= -{}^{K}\!/_{2}\, var(Y_i) \\ &= cov\left[X_{i-1}, \phi(\theta - X_{i-1})\right] \\ &= -\frac{\sigma^2}{\sqrt{1+\sigma^2}} \frac{1}{\sqrt{2\pi}} \end{aligned} \tag{12}$$

These results are obtained using collecting squares see Appendix C for the equations and theorem (4) for the same result by using skew-normal distributions. Molenaar (2007, p. 10–12) obtained the same result using a different approach, namely, orthogonal rotations.

Theorem 4. *Normalized product of normal density and normal distribution*

$$\int_{-\infty}^{\infty} \phi(z) z \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{z^2}{2\sigma^2}\right) dz = \frac{\sigma^2}{\sqrt{1+\sigma^2}} \frac{1}{\sqrt{2\pi}}$$

Proof. Proof is given by Azzalini (1985, p.174), which shows that the product of a normal density and a normal distribution results in a *skew-normal distribution* (e.g., Azzalini & Dalla Valle, 1996; Azzalini & Capitanio, 1999). □

It only remains to solve formula (12), of which the variance component has not been worked out yet. Using the variance decomposition from theorem

(3), we can write

$$\begin{aligned}
var(Y_i) &= var\left[\mathcal{E}(Y_i|X_{i-1})\right] + \mathcal{E}\left[var(Y_i|X_{i-1})\right] \\
&= var\left[\phi(\theta - X_{i-1})\right] + \mathcal{E}\left[var(Y_i|X_{i-1})\right] \\
&= \mathcal{E}\left[\phi(\theta - X_{i-1})^2\right] - \mathcal{E}\left[\phi(\theta - X_{i-1})\right]^2 \\
&+ \mathcal{E}\left[\phi(\theta - X_{i-1})\right] - \mathcal{E}\left[\phi(\theta - X_{i-1})^2\right] \\
&= \mathcal{E}\left[\phi(\theta - X_{i-1})\right] - \mathcal{E}\left[\phi(\theta - X_{i-1})\right]^2 \\
&\Updownarrow \text{ which are Bernoulli distributed } \pi - \pi^2 = \pi(1-\pi) \\
&= \mathcal{E}\left[\phi(\theta - X_{i-1})\right] - \left\{1 - \mathcal{E}\left[\phi(\theta - X_{i-1})\right]\right\} \\
&= {}^1\!/_2\left(1 - {}^1\!/_2\right) = {}^1\!/_4
\end{aligned}$$

With this result, equation (12) can now be rewritten as follows

$$\begin{aligned}
-\frac{\sigma^2}{\sqrt{1+\sigma^2}}\frac{1}{\sqrt{2\pi}} &= -{}^K\!/_8 \\
&\Updownarrow \\
K &= \frac{8\sigma^2}{\sqrt{2\pi(1+\sigma^2)}}.
\end{aligned}$$

From the calculations in this section, one can observe that K is independent of θ. If K had been dependent, it would have to be estimated along with X_i as in Kalman filtering (e.g., Kalman, 1960; van Rijn, 2008). The relationship between Kalman filtering and the ERS is that Kalman filters consist of a *predict* and an *update* step and adaptive ERS consist of an *item selection* and also an *update* step. One should note that K and σ^2 are dependent, where one of them can be chosen freely. Now that both the mean and the variance are found to be stationary, the stationarity of the distribution will be evaluated next.

2.6.1. *Stationarity of the distribution.* For tracktabiltiy purposes, it has so far been assumed that $X_{i-1} \sim \mathcal{N}\left(\theta, \sigma^2\right)$. If this assumption is true, it needs to be investigated whether also $X_i \sim \mathcal{N}\left(\theta, \sigma^2\right)$. The following equation rewrites the probability of obtaining a certain score on X_i into two skew-normal distributions

$$\begin{aligned}
\Pr(X_i \le c) &= \Pr(X_i \le c|z_i = 1)\Pr(z_i = 1) \\
&+ \Pr(X_i \le c|z_i = 0)\Pr(z_i = 0) \\
&= \Pr\left[X_{i-1} + K\left(z_i - 1/2\right) \le c|z_i = 1\right] 1/2 \\
&+ \Pr\left[X_{i-1} + K\left(z_i - 1/2\right) \le c|z_i = 0\right] 1/2 \\
&= \Pr\left(X_{i-1} + K/2 \le c|z_i = 1\right) 1/2 \\
&+ \Pr\left(X_{i-1} - K/2 \le c|z_i = 0\right) 1/2 \\
&= \Pr\left(X_{i-1} \le c - K/2|z_i = 1\right) 1/2 \\
&+ \Pr\left(X_{i-1} \le c + K/2|z_i = 0\right) 1/2
\end{aligned}$$

After which it logically follows that

$$\begin{aligned}
\Pr(X_i \le c) &= \int_{-\infty}^{c-K/2} \phi(\mu - x)\frac{1}{\sqrt{2\pi\sigma}} \exp\left[\frac{-(x-\mu)^2}{2\sigma^2}\right] dx \\
&+ \int_{-\infty}^{c+K/2} \left[1 - \phi(\mu - x)\right] \frac{1}{\sqrt{2\pi\sigma}} \exp\left[\frac{-(x-\mu)^2}{2\sigma^2}\right] dx \\
&\neq \phi\left(\frac{c-\mu}{\sigma}\right),
\end{aligned}$$

which shows that the assumption of a normal distributed X_{i-1} does not lead to a normal distributed X_i. Thus while both the stationarity of the mean and of the variance are guaranteed, the normal distribution is not the invariant distribution in the adaptive normal ogive rating system. A solution is provided in the following section where the ERS as a mixture distribution is introduced.

2.7. **Mixture distributed Elo rating system.** In the adaptive normal ogive variant of the ERS, a stationary solution is not obtained with an invariant normal distribution. A mixture of two normal distributions, or *Mixtures of Multivariate Normal Distributions* (MMND) model, is considered. Two response distributions are assumed, one for correct and one for incorrect responses

$$\begin{aligned}
X|Y = 1 &\sim \mathcal{N}(\mu - a, \sigma^2) \\
X|Y = 0 &\sim \mathcal{N}(\mu + a, \sigma^2)
\end{aligned}$$

with the parameter a facilitating the shift between the two. The probability of a correct response is

$$\Pr(Y = 1) = \pi.$$

Efron (1975) and Kiefer (1980) have already indicated the relation between the logit model, MMND, and the *Normal Discriminant Analsysis* (NDA). This is pointed out by Sapra (1991, p.266), who uses Bayes' theorem to prove how NDA and MMND models also imply the logit model, resulting in

$$\Pr(Y = 1|X = x) = \frac{\exp\left[\frac{2a(\mu-x)}{\sigma^2}\right]\frac{\pi}{1-\pi}}{1+\exp\left[\frac{2a(\mu-x)}{\sigma^2}\right]\frac{\pi}{1-\pi}}. \tag{13}$$

which has as a density of the mixture distribution $m(x)$

$$m(x) = m(x|Y = 1)\pi + m(x|Y = 0)(1 - \pi).$$

Judging from equation (13), one can theoretically offer the respondent an item with a certain difficulty such that a normal distribution of occasions with expectation μ and variance σ^2 is obtained. The system is therefore adaptive, with as goal to select specific items to obtain a stationairy distribution. This formula is a so-called *match maker rule*, which adaptively selects the exact item to offer the respondent.

$$\begin{aligned} x &= x_i + \frac{\sigma^2}{2a}\ln\left(\frac{\pi}{1-\pi}\right) \\ &\Updownarrow \text{ using equation (13)} \\ \Pr(Y = 1|X = x) &= \frac{\exp\left[\frac{x-\mu}{\sigma^2}\right]}{1+\exp\left[\frac{x-\mu}{\sigma^2}\right]} \end{aligned}$$

For now, it is assumed that the item with the exact difficulty that this match maker rule provides is available. This is a limiting assumption, suggesting an infinite amount of items with known parameters available. Though in this framework it seems also possible to estimate item parameters along, its implications are left open for further research.

It was recognized that part of the rating system consist of a match maker rule, the other part can be identified as an *update rule*. The update rule corresponding with this specific match maker rule is presented below.

$$\begin{aligned} X_i &= X_{i-1} + 2a\left(Y_i - 1/2\right) \\ &= X_{i-1} + (2Y_i - 1)a. \end{aligned} \tag{14}$$

Starting with a mixture of normal distributions, the match maker and update rule provide a known normal distribution. It is noted that in order to obtain the correct mixture distribution again, this normal distribution needs to be shifted occasionally. Here one can recognize a similarity to the Metropolis-Hastings algorithms (Hastings, 1970), where an update is occasionally accepted to obtain a specific distribution. The advantage of this step is that the correct mixture distribution is obtained. A disadvantage is that the ability estimate is not always updated after a match. To facilitate the occasional shift, a Bernoulli distributed variable Z_i is included with parameter $1 - \pi$ and independent of both X_{i-1} and Y_i. The update rule providing the mixture of normal distributions then takes the following shape

$$X_i = X_{i-1} + (2Y_i - 1)a + (2Z_i - 1)a.$$

The process described above to obtain the correct mixture distribution can also be presented graphically, as in Figure (7). Several steps are illus-

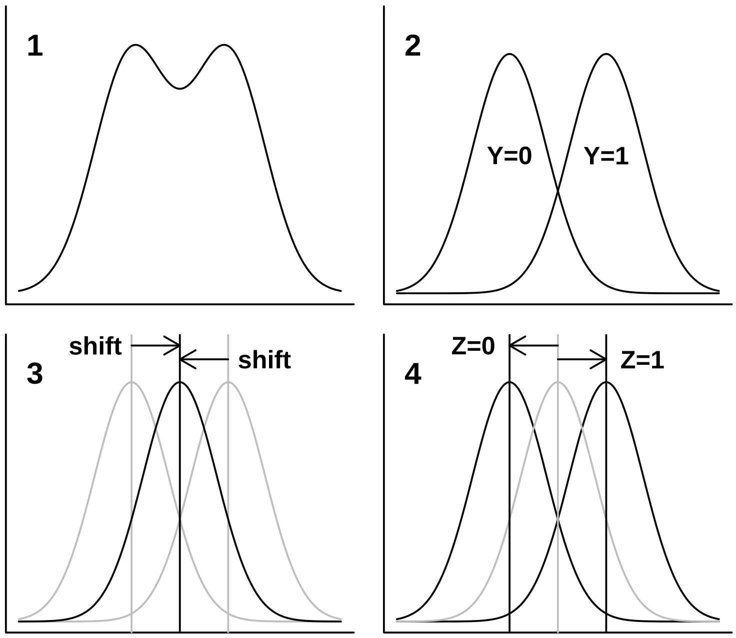

FIGURE 7. Composing the correct mixture distribution

trated, starting with the unknown mixture distribution in step 1. In step 2, the answer Y determines which normal distribution is appropriate. In step 3, these distributions are shifted toward a single normal distribution by applying a specific item, as explained in equation (14). The final step consist of resampling according to the Bernoulli distributed variable Z_i, to obtain the correct mixture distribution. To further illustrate this process, a scheme is provided in Figure (8) that describes the necessary steps. The scheme consists of two columns, one identified with $\mathcal{M}$ for mixture distributed states, and one with $\mathcal{N}$ for normal distributed states. Each of these columns can be regarded as interdependant Markov chains, one with a mixture and one with a normal distribution, indicated by vertical dashed arrows. One can also read the scheme in a similar manner as Figure (7), indicated by solid arrows. Starting with a mixture distribution state X_1, answer Y_1 to the

$$
\begin{array}{ccccc}
 & & \mathcal{M} & & \mathcal{N} \\
 & & X_1 & \xrightarrow{Y_1} & X_1^* = X_1 + (2Y_1 - 1)a \\
 & & \downarrow & \overset{Z_1}{\swarrow} & \downarrow \\
X_1^* + (2Z_1 - 1)a & = & X_2 & \xrightarrow{Y_2} & X_2^* \\
 & & \downarrow & \overset{Z_2}{\swarrow} & \downarrow \\
 & & X_3 & & X_3^*
\end{array}
$$

FIGURE 8. Scheme of two Markov processes

selected item provides a normal distributed X_1^*. To obtain a mixture distribution again, Z_1 is used to shift the normal distribution to the mixture state X_2. Thus, although a mixture distributed Markov chain was created as stationary distibution, using this scheme one can as well depart from a normal distributed Markov chain.

Using these interdependent Markov chains, a stationary distribution of scores around the true ability has been achieved. It has already been noted that there is a disadvantage in not always facilitating an update of the estimate, and that there are some limiting assumptions concerning availability of items with estimated parameters.

3. Results

In the present study the properties of several variants of the ERS where investigated. Unexpected was the lack of stationarity of the mean in the non-adaptive logistic ERS. A normal ogive variant was investigated that turned out to lead to complicated expressions for even the variance of the stationary distribution. It was expected that even less tractable results would be obtained for the stationary distribution itself, which is why this route was not explored further. However, it may still prove a useful direction for further research. The properties of the adaptive variants of the ERS seemed more attractive, with stationarity of the mean shown to hold for a general adaptive ERS. Stationarity of the variance was found to be intractable for the logistic variant, and stationarity of the distribution was lacking for the normal ogive adaptive ERS. A combination of the different variants proved to hold the right properties. Adaptive, symmetric, and logistic elements were combined in a mixture ERS. The mixture variant obtains a stationary mixture distribution, with as drawbacks a possible decline of parameter updates, and the introdution of several new assumptions. Despite these drawbacks, the algorithm seems suitable for analysis and for further research. Possible implementations will be discussed in the next section.

4. Discussion

The investigation of the properties of several variants of the ERS, resulted in some problems and solutions. First, the implications of the assumptions are discussed. Second, the impact of the biased logistic non-adaptive ERS is assessed. Third, the ERS is compared with a traditional IRT approach to the data under consideration. The discussion ends with possible practical implementations of the ERS.

Throughout this paper, several assumptions are made. In all the adaptive variants of the ERS presented, item parameters are assumed to be known since an item is selected with a difficulty level approximately to the respondent's ability level. However, this assumption is not necessarily a major obstacle. Item parameters can, for example, be pre-tested, and moreover, the algorithm allows for estimating item parameters alongside the person parameters. An assumption that does not hold in practice is the assumed static abilities, e.g., no learning takes place. The algorithm is very suitable to estimate changing ability. No static growth model is assumed and, owing to its erratic update steps, the algorithm can quickly follow changing trends. An evaluation of the properties under the assumption of growth is still an open research question.

Another remark has to be made concerning the bias found in the non-adaptive ERS. In practical implementations, this bias might have serious consequences. However, for sports, it seems the effect could be rather limited. It is quite natural for equally able teams or players to compete. Chess tournaments are organized for certain ability levels and many other sports have ability organized leagues. The implementations of ERS are quite adaptive, for which a stationary mean was found. Hence, consequences might be less serious in practical implementations than expected from the bias found in the non-adaptive ERS.

A more traditional IRT approach to the data being evaluated is calculating *maximum likelihood* (ML) estimates. The ERS under consideration provides several advantages over ML estimation. ML estimates are updated for all persons and items after a single question is answered by a single person, which is conceptually difficult to defend to individual players and computationally demanding in larger applications. Furthermore, assuming a growth model is often needed in ML estimation with the risk of introducing bias. The ERS does not have these limitations and therefore seems a suitable alternative to the data as discussed in this research.

As mentioned in the introduction, a possible implementation of this research would be in one of Cito's monitoring products, namely the *leerling -en onderwijsvolgsysteem (LOVS)*, or student monitoring system. The LOVS usually administers two to three tests each year, to which the proposed method might add intermediate ability tracking. The advantage of this tandem is that the occasional lengthier tests provide reliable estimates, which can serve as calibrations of the high frequency Elo estimates obtained between administrations. Another interesting implementation is on data from the *Rekentuin*. The *Rekentuin* is an arithmetic testing web site developed by the University of Amsterdam. Pupils from primary education can answer arithmetic questions, which causes a virtual garden to grow on their screens. Pupils access the *Rekentuin* website from their schools and from home, creating varying amounts of data on varying time points. These data are already analyzed using the Elo rating system, and might be extended with the findings presented in this report. Further research in needed to investigate whether other variants of the rating system can also fully control the distributions of ratings with less strict assumptions.

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Appendices

A. **Two updates in a non-adaptive logistic Elo rating system.** Stationarity of the mean was explored by writing out two updates of the rating system. It is shown that it does not hold, even considering the simple situation where the starting value is exactly the true ability, thus $\mu(\theta) = \mu(x_0) = \mu$. For the first trial, summing up both possible Y_1 responses multiplied by their binomial probabilities

$$\begin{aligned}
\mathcal{E}(X_1|X_0 = x) &= (\mu)^1(1-\mu)^0\,[x + K(1-\mu)] \\
&+ (\mu)^0(1-\mu)^1\,[x + K(0-\mu)] \\
&= \mu(x - K\mu) + K\mu + (x - K\mu) - \mu(x - K\mu) \\
&= x \qquad \text{(A–1)}
\end{aligned}$$

For the second trial, summing up the outcomes from the first trial, for both a correct and an incorrect answer

$$\begin{aligned}
\mathcal{E}(X_2|X_0 = x) &= (\mu)^2(1-\mu)^0\left[x + K(1-\mu) + K\left(1 - \frac{\exp[x + K(1-\mu)]}{1 + \exp[x + K(1-\mu)]}\right)\right] \\
&+ (\mu)^1(1-\mu)^1\left[x + K(1-\mu) + K\left(0 - \frac{\exp[x + K(1-\mu)]}{1 + \exp[x + K(1-\mu)]}\right)\right] \\
&+ (\mu)^1(1-\mu)^1\left[x + K(0-\mu) + K\left(1 - \frac{\exp[x + K(0-\mu)]}{1 + \exp[x + K(0-\mu)]}\right)\right] \\
&+ (\mu)^0(1-\mu)^2\left[x + K(0-\mu) + K\left(0 - \frac{\exp[x + K(0-\mu)]}{1 + \exp[x + K(0-\mu)]}\right)\right]
\end{aligned}$$

which can be written out as follows

$$
\begin{aligned}
&\mu^2\left[x + K(1-\mu) + K\left(1 - \frac{\exp[x+K(1-\mu)]}{1+\exp[x+K(1-\mu)]}\right)\right] \quad + \\
&\mu\left[x + K(1-\mu) + K\left(0 - \frac{\exp[x+K(1-\mu)]}{1+\exp[x+K(1-\mu)]}\right)\right] \quad + \\
-&\mu^2\left[x + K(1-\mu) + K\left(0 - \frac{\exp[x+K(1-\mu)]}{1+\exp[x+K(1-\mu)]}\right)\right] \quad + \\
&\mu\left[x + K(0-\mu) + K\left(1 - \frac{\exp[x+K(0-\mu)]}{1+\exp[x+K(0-\mu)]}\right)\right] \quad + \\
-&\mu^2\left[x + K(0-\mu) + K\left(1 - \frac{\exp[x+K(0-\mu)]}{1+\exp[x+K(0-\mu)]}\right)\right] \quad + \\
&\left[x + K(0-\mu) + K\left(0 - \frac{\exp[x+K(0-\mu)]}{1+\exp[x+K(0-\mu)]}\right)\right] \quad + \\
-&2\mu\left[x + K(0-\mu) + K\left(0 - \frac{\exp[x+K(0-\mu)]}{1+\exp[x+K(0-\mu)]}\right)\right] \quad + \\
&\mu^2\left[x + K(0-\mu) + K\left(0 - \frac{\exp[x+K(0-\mu)]}{1+\exp[x+K(0-\mu)]}\right)\right]
\end{aligned}
$$

collecting terms, one obtains

$$
\begin{aligned}
&x + K\mu\left[1 + \frac{\exp(x-K\mu)}{1+\exp(x-K\mu)} - \frac{\exp(x+K-K\mu)}{1+\exp(x+K-K\mu)}\right] - K\frac{\exp(x-K\mu)}{1+\exp(x-K\mu)} = \\
&x - K\left[\mu\frac{\exp(x+K-K\mu)}{1+\exp(x+K-K\mu)} + (1-\mu)\frac{\exp(x-K\mu)}{1+\exp(x-K\mu)} - \mu\right].
\end{aligned}
$$

One can observe that a solution is obtained with $K = 0$. The numerator is simplified and equated to zero to inspect whether other solutions exist

$$
\begin{aligned}
(\mu-1)\exp(x-K\mu+K) - \mu\exp(x-K\mu) - 1 &= 0 \\
&\Updownarrow \\
(\mu-1)\exp(x-K\mu+K) &= 1 + \mu\exp(x-K\mu) \\
&\Updownarrow \\
\frac{(\mu-1)\exp(x-K\mu+K)}{1+\mu\exp(x-K\mu)} &= 1 \\
&\Updownarrow \\
\frac{\mu-1}{\mu}\exp(K)\frac{\exp(x-K\mu+\ln\mu)}{1+\exp(x-K\mu+\ln\mu)} &= 1 \\
&\Updownarrow \\
-\frac{\exp(K)}{\exp(x)}\frac{\exp(x-K\mu+\ln\mu)}{1+\exp(x-K\mu+\ln\mu)} &= 1.
\end{aligned}
$$

Since the left fraction of the last result is negative and the right fraction positive, it is clear that no additional solutions exists for any parameters.

B. **Two updates in an adaptive logistic Elo rating system.** While stationarity of the mean has already been shown to hold for the symmetrical adaptive ERS, here it is worked out for two updates for the variant with a logistic CDF. As demonstrated for the logistic ERS in equation (A–1), the first update leads to a correct mean regardless of μ, which is repeated below for the adaptive ERS

$$\begin{aligned}\mathcal{E}(X_1|X_0=\theta) &= \frac{\exp(\theta-\theta)}{1+\exp(\theta-\theta)}\left[\theta+K\left(1-{}^1\!/\!{}_2\right)\right] \\ &+ \frac{\exp(\theta-\theta)}{1+\exp(\theta-\theta)}\left[\theta+K\left(0-{}^1\!/\!{}_2\right)\right] \\ &= {}^1\!/\!{}_2\,(\theta+{}^K\!/\!{}_2)+{}^1\!/\!{}_2\,(\theta-{}^K\!/\!{}_2) \\ &= \theta\end{aligned}$$

It is shown below that this also holds for the second iteration, with $\mu = {}^1\!/\!{}_2$. For the second iteration, summing up the outcomes from the first iteration, for both a correct and an incorrect answer in

$$\begin{aligned}&\frac{\exp(\theta-\theta)}{1+\exp(\theta-\theta)}\frac{\exp\left[\theta-(\theta+{}^K\!/\!{}_2)\right]}{1+\exp\left[\theta-(\theta+{}^K\!/\!{}_2)\right]}\left[\theta+{}^K\!/\!{}_2+K\left(1-{}^1\!/\!{}_2\right)\right] \quad + \\ &\frac{\exp(\theta-\theta)}{1+\exp(\theta-\theta)}\left\{1-\frac{\exp\left[\theta-(\theta+{}^K\!/\!{}_2)\right]}{1+\exp\left[\theta-(\theta+{}^K\!/\!{}_2)\right]}\right\}\left[\theta+{}^K\!/\!{}_2+K\left(0-{}^1\!/\!{}_2\right)\right] \quad + \\ &\left[1-\frac{\exp(\theta-\theta)}{1+\exp(\theta-\theta)}\right]\frac{\exp\left[\theta-(\theta+{}^K\!/\!{}_2)\right]}{1+\exp\left[\theta-(\theta+{}^K\!/\!{}_2)\right]}\left[\theta+{}^K\!/\!{}_2+K\left(1-{}^1\!/\!{}_2\right)\right] \quad + \\ &\left[1-\frac{\exp(\theta-\theta)}{1+\exp(\theta-\theta)}\right]\left\{1-\frac{\exp\left[\theta-(\theta+{}^K\!/\!{}_2)\right]}{1+\exp\left[\theta-(\theta+{}^K\!/\!{}_2)\right]}\right\}\left[\theta+{}^K\!/\!{}_2+K\left(0-{}^1\!/\!{}_2\right)\right]\end{aligned} \tag{B–2}$$

where $\frac{\exp(-{}^K\!/\!{}_2)}{1+\exp(-{}^K\!/\!{}_2)} \equiv \epsilon$ and accordingly $\frac{\exp({}^K\!/\!{}_2)}{1+\exp({}^K\!/\!{}_2)} \equiv 1-\epsilon$

$$ {}^1\!/\!{}_2\left[\epsilon(\theta+K)+(1-\epsilon)\theta+(1-\epsilon)\theta+\left[1-(1-\epsilon)\right](\theta-K)\right]=\theta$$

These equations prove that the expected value remains at the true value after two iterations when starting at the true value. One should note that the symmetrical result of the response function in equation (B–2) is of key importance, discussed by Batchelder et al. (1992).

C. **Collecting squares.**

$$
\begin{aligned}
cov\left[X_{i-1}, F(\theta - X_{i-1}\right] &= cov\left[X_{i-1}, \Phi(\theta - X_{i-1})\right] \\
&= \mathcal{E}\left[X_{i-1}, \Phi(\theta - X_{i-1})\right] - \mathcal{E}\left(X_{i-1}\right)\mathcal{E}\left[\Phi(\theta - X_{i-1})\right] \\
&= \mathcal{E}\left[X_{i-1}, \Phi(\theta - X_{i-1})\right] - \theta/2 \\
&= \int_{-\infty}^{\infty} x\Phi(\theta - x)x\phi\left(x|\theta, \sigma^2\right)dx - \theta/2
\end{aligned}
$$

and a transformation is used to remove θ from the IRF

$$z = \theta - x \Leftrightarrow x = \theta - z$$

which results in

$$
\begin{aligned}
cov[X_{i-1}, \Phi(\theta - X_{i-1})] &= \int_{-\infty}^{\infty} \Phi(z)(\theta - z)\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{z^2}{2\sigma^2}\right)dz - \theta/2 \\
&= \frac{\theta}{2} - \int_{-\infty}^{\infty} \Phi(z)z\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{z^2}{2\sigma^2}\right)dz - \theta/2 \\
&= -\int_{-\infty}^{\infty}\int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{y^2}{2}\right)z\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{z^2}{2\sigma^2}\right)dydz \\
&\quad \text{redefining the definite integral } y \le z \Leftrightarrow w = y - z \le 0 \\
&\Rightarrow -\int_{-\infty}^{\infty}\int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(w+z)^2}{2}\right)z\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{z^2}{2\sigma^2}\right)dwdz
\end{aligned}
$$

collecting squares

$$
\begin{aligned}
-\frac{(w+z)^2}{2} - \frac{z^2}{2\sigma^2} &= -\frac{(w+z)^2\sigma^2 + z^2}{2\sigma^2} \\
&= -\frac{w^2\sigma^2 + 2wz\sigma^2 + z^2\sigma^2 + z^2}{2\sigma^2} \\
&= -\frac{w^2\sigma^2 + (1+\sigma^2)\left(z^2 + 2\frac{\sigma^2}{1+\sigma^2}\right)}{2\sigma^2} \\
&= -\frac{w^2\sigma^2 + (1+\sigma^2)\left[\left(z + w\frac{\sigma^2}{1+\sigma^2}\right)^2 - \left(w\frac{\sigma^2}{1+\sigma^2}\right)^2\right]}{2\sigma^2} \\
&= -\frac{\sigma^2\frac{w^2}{1+\sigma^2}}{2\sigma^2} - \frac{\left[z - \left(-w\frac{\sigma^2}{1+\sigma^2}\right)\right]^2}{\frac{1}{1+\sigma^2}2\sigma^2}
\end{aligned}
$$

which can be split again in exponents

$$
\begin{aligned}
& -\int_{-\infty}^{\infty}\int_{-\infty}^{0}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{\sigma^2\frac{w^2}{1+\sigma^2}}{2\sigma^2}\right)\frac{1}{\sigma\sqrt{2\pi}} \\
& \exp\left[\frac{\left(z+w\frac{\sigma^2}{1+\sigma^2}\right)^2}{\frac{1}{1+\sigma^2}2\sigma^2}\right]zdwdz \\
= \; & -\int_{-\infty}^{\infty}\int_{-\infty}^{0}\frac{1}{\sqrt{1+\sigma^2}\sqrt{2\pi}}\exp\left(-\frac{1}{2}\frac{w^2}{1+\sigma^2}\right)\frac{1}{\sigma^*\sqrt{2\pi}} \\
& \exp\left[-\frac{1}{2(\sigma^*)^2}\left(z-\mu^*\right)^2\right]zdwdz \\
& \text{where } \sigma^* = \sqrt{\frac{\sigma^2}{1+\sigma^2}} \text{ and } \mu^* = -w\left(\sigma^*\right)^2 \\
= \; & -\int_{-\infty}^{0}\left\{\int_{-\infty}^{\infty}z\frac{1}{\sigma^*\sqrt{2\pi}}\exp\left[-\frac{1}{2(\sigma^*)^2}\left(z-\mu^*\right)^2\right]dz\right\}\frac{1}{\sqrt{1+\sigma^2}\sqrt{2\pi}} \\
& \exp\left(-\frac{1}{2}\frac{w^2}{1+\sigma^2}\right)dw
\end{aligned}
$$

where the integral between square brackets can be recognized as the expected value $\mathcal{E}(z) = \mu^*$

$$
\begin{aligned}
= \; & -\int_{-\infty}^{0}\left[-w\frac{\sigma^2}{1+\sigma^2}\right]\frac{1}{\sqrt{1+\sigma^2}\sqrt{2\pi}}\exp\left(-\frac{1}{2}\frac{w^2}{1+\sigma^2}\right)dw \\
= \; & \frac{\sigma^2}{1+\sigma^2}\int_{-\infty}^{0}w\frac{1}{\sqrt{1+\sigma^2}\sqrt{2\pi}}\exp\left(-\frac{1}{2}\frac{w^2}{1+\sigma^2}\right)dw
\end{aligned}
$$

where remaining integral can be rewritten as the expected value of a half normal CDF

$$
\begin{aligned}
\text{(C–3)} \qquad = \; & \frac{\sigma^2}{1+\sigma^2}\int_{-\infty}^{0}w\frac{1}{\sigma^*\sqrt{2\pi}}\exp\left[-\frac{1}{2\left(\sigma^*\right)^2}\left(w-\mu^*\right)^2\right]dw \\
& \text{where } \sigma^* = \sqrt{1+\sigma^2} \text{ and } \mu^* = 0
\end{aligned}
$$

It is shown that this result is similar to the differentiated normal distribution over μ

$$\begin{aligned}
F(y) &= \int_{-\infty}^{y} \frac{1}{\sigma^*\sqrt{2\pi}} \exp\left[-\frac{1}{2(\sigma^*)^2}(w-\mu^*)^2\right] dw \\
&\Updownarrow \\
f(y) &= \int_{-\infty}^{y} \frac{1}{\sigma^*\sqrt{2\pi}} \frac{w-\mu^*}{(\sigma^*)^2} \exp\left[-\frac{1}{2(\sigma^*)^2}(w-\mu^*)^2\right] dw \\
&= \frac{\mu^*}{(\sigma^*)^2} f(y) - \frac{1}{(\sigma^*)^2} \\
&\quad \int_{-\infty}^{y} w \frac{1}{\sigma^*\sqrt{2\pi}} \exp\left[-\frac{1}{2(\sigma^*)^2}(w-\mu^*)^2\right] dw \\
&\Updownarrow \\
\mu^* F(y) - (\sigma^*)^2 f(y) &= \int_{-\infty}^{y} w \frac{1}{\sigma^*\sqrt{2\pi}} \exp\left[-\frac{1}{2(\sigma^*)^2}(w-\mu^*)^2\right] dw
\end{aligned}$$

where the last integral has the desired shape of the integral in equation (C–3), where $y = \mu^* = 0$

$$\begin{aligned}
-(\sigma^*)^2 \frac{1}{\sigma^*\sqrt{2\pi}} &= \int_{-\infty}^{y} w \frac{1}{\sigma^*\sqrt{2\pi}} \exp\left[-\frac{1}{2(\sigma^*)^2}(w-\mu^*)^2\right] dw \\
&= -\frac{1+\sigma^2}{\sqrt{1+\sigma^2}} \frac{1}{\sqrt{2\pi}} \text{ since } \sigma^* = \sqrt{1+\sigma^2}
\end{aligned}$$

which is multiplied by the constant in front of the integral in equation (C–3) to obtain a solution

$$\begin{aligned}
cov(X_{i-1}, Y_i) &= -\frac{1+\sigma^2}{\sqrt{1+\sigma^2}} \frac{1}{\sqrt{2\pi}} \frac{\sigma^2}{1+\sigma^2} \\
&= -\frac{\sigma^2}{\sqrt{1+\sigma^2}} \frac{1}{\sqrt{2\pi}}
\end{aligned}$$

The result of the derivations above are repeated in theorem (4).

D. **Inverse cumulative distribution function transformation.**

$$P(X \le x) = F(x) \Rightarrow P[F(X) \le x] = P\left[X \le F^{-1}(u)\right] \Rightarrow F\left[F^{-1}(u)\right] = u$$

where $u = G(x)$ such that $x = G^{-1}(u)$

$$\int_{-\infty}^{\infty} F(x)g(x)dx = \int_0^1 F\left[G^{-1}(u)\right] g\left[G^{-1}(u)\right] du = \int_0^1 F\left[G^{-1}(u)\right] du.$$

The symmetry of $G^{-1}(u)$ can be derived from the symmetry of $G(x)$

$$G(x) = 1 - G(-x) = u \Rightarrow G(-x) = 1 - u \Rightarrow x = -G^{-1}(1-u) = G^{-1}(u)$$

from which follows

$$\begin{aligned}
\int_0^1 F\left(G^{-1}(u)\right) du &= \int_0^1 1 - F\left[-G^{-1}(u)\right] du \\
&= \int_0^1 1 - F\left[G^{-1}(1-u)\right] du \\
&= 1 - \int_0^1 F\left[G^{-1}(1-u)\right] du \\
&= 1 - \int_0^1 F\left[G^{-1}(v)\right] dv, \text{ where } v = 1-u
\end{aligned}$$

the only solution where the first result and the last result are equal is

$$\int_0^1 F\left[G^{-1}(u)\right] du = 1 - \int_0^1 F\left[G^{-1}(u)\right] du \Rightarrow \int_0^1 F\left[G^{-1}(u)\right] du = 1/2.$$